Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

SOME CIRCLES ASSOCIATED WITH CONCYCLIC POINTS

By J. L. Coolidge

If a system of any number of lines, n, be given in a plane, they may be associated with n circles or n points in a variety of ways. The first of these was discovered by Clifford, and gives rise to the following theorem: *

Let n lines be given in a plane, no two of which are parallel. If n be even we may associate with them a point, and if n be odd a circle, in such a way that the point is common to the n circles each associated with a set of $n-1$ *lines obtained by omitting each of the given lines in turn; the circle contains the n points each associated with a set of* $n-1$ *lines in the same way.*

A more remarkable theorem of the same sort was discovered in 1891 by Pesci; it may be stated as follows: †

Let n lines be given in a plane, no two of which are parallel. We may associate with them a point and a circle in such a way that the point is common to the n circles each associated with a set of $n-1$ *lines obtained by omitting each of the given lines in turn, while the circle contains the centres of these n circles. When* $n > 4$ *the point will not lie on the circle.*

In the special case, $n = 4$, the circles are those circumscribed to the four triangles formed by the four lines. These circles pass through the focus of the parabola which touches the four lines, and their centres lie on a circle through that focus. This theorem is due to Steiner. ‡

There is one more theorem of this same sort which was given by Grace, and may be stated as follows: §

Let n lines be given in a plane, no two of which are parallel, and n concyclic points, one on each of them. If n be odd we may associate with them a point, and if n be even a circle, in such a way that the point is common to the n circles each associated with a set of $n-1$ *lines obtained by omitting each of the given lines in turn; the circle contains the n points each associated with a set of* $n-1$ *lines in the same way.*

---

[*] A synthetic Proof of Miquel's Theorem, *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1870. At least two subsequent writers seem to have rediscovered it.

[†] Dei cercoli circonscritti ai triangoli formati di n rette in un piano: *Periodico di Matematica*, vol. 5, 1891.

[‡] See his *Collected Works*, vol. 1, p. 223.

[§] Circles, Spheres and Linear Complexes, *Transactions of the Cambridge Philosophical Society*, vol. 16, 1898.

When the circle of the n points reduces to a line, Grace's theorem reduces to Clifford's.

The first theorem which I wish to develop in the present paper is analogous to both Pesci's and Grace's, and may be stated as follows:*

I) *Given n points on a circle where $n \geqq 4$. We may associate with them a point and a circle in the following manner:*

a) *the point is the centre of the circle;*

b) *the radius of the circle is one half that of the given circle;*

c) *the point lies on the n circles each associated with a system of $n-1$ concyclic points obtained by omitting each of the given points in turn;*

d) *the circle contains the centres of these n circles.*

The method of proof is substantially that which is used to advantage for Clifford and Grace's theorems, and depends upon the two following simple theorems:

1) *If a triangle be given, and a point be marked on each side, the three circles, each of which passes through a vertex and the two marked points of the adjacent sides are concurrent.*†

We see, in fact, that each circle and the opposite side of the triangle constitute a cubic curve, and as these three cubics have eight common points, they have a ninth point in common also. The theorem is also easily proved by elementary geometry. If we invert this figure with regard to any convenient circle, we shall get

2) *Given a circular triangle formed by three circles through a common point. If a point be marked on each side of the triangle, the three circles, each of which passes through a vertex and the two marked points of the adjacent sides, are concurrent.*

Let us give one corollary to the last theorem which will be of use to us in the present paper, though not for our immediate purpose:

3) *Given four points upon a circle arranged in cyclic order, and four circles of arbitrary radius each connecting a successive pair of points. Then the remaining intersections of successive pairs of circles lie on another circle.*‡

---

* Since the present article went to press I have found the analog of this theorem in three dimensions, though not in so many words in an article by Intrigila, "Sul tetrædro." *Rend. Accad. Sci. di Napoli*, vol. 22, 1883. It is the more remarkable that the present theorem should seem to be new.

† Miquel, Théorèmes de Géometrie, *Liouville's Journal*, vol. 3, 1838.

‡ Miquel, "Théorèms de Géometrie," *Liouville's Journal*, vol. 9, 1844.

The four points shall be called $P_1\ P_2\ P_3\ P_4$, the circle upon which they lie c, while c_{ij} is the circle through P_i and P_j. The circles c_{ij} and c_{jk} shall intersect in P_j and P'_j.

The circles $c\ c_{34}\ c_{41}$ are concurrent in P_4. They form a circular triangle whose vertices are $P_1\ P_3\ P_4$. The side c contains also P_2, the side c_{34} contains P'_3, and the side C_{41} contains P'_1. On the other hand $P_1\ P'_1\ P_2$ lie upon c_{12} and $P_3\ P'_3\ P_2$ lie upon c_{23}. These intersect again in P'_2; hence, by 2) P'_2 lies on the circle through $P'_4\ P'_1\ P'_3$.

We may now proceed to prove our Theorem I:

When $n = 2$ we shall associate with the given points the point mid way between them. When $n = 3$ we shall associate with the given points their nine-point circle and its centre. Notice that this circle passes through the three points associated with each pair of the given points. We may, without restriction, assume that the radius of the circumscribed circle is equal to 2. When $n = 4$ we have a well-known theorem whereby the nine-point circles of the four triangles formed by four concyclic points are concurrent.*

We see, in fact, that all conics of the pencil through the vertices and ortho-centre of a triangle must be rectangular hyperbolas, for the involution which they determine on the line at infinity has three mutually perpendicular directions. The locus of the centres of these conics is a conic through the double points of this involution, which are the circular points at infinity, and the vertices of the common self-conjugate triangle, which are the feet of the altitudes. In other words, the nine-point circle is the locus of the centres of all conics through the three vertices of a triangle and the ortho-centre, and if four points be given, which are not the vertices and ortho-centre of a triangle, the nine-point circles of their four triangles are concurrent in the centre of the rectangular hyperbola through the four points. When, further, the points are concyclic, their four nine-point circles have all the same radius, and so their centres lie on a circle of equal radius around the point of concurrence as centre. Our theorem is thus established in the case $n = 4$.

Let us pass on to the case $n = 5$. Here we shall have five concyclic points $P_1\ P_2\ P_3\ P_4\ P_5$.

The circle associated with $P_j\ P_k P_l\ P_m$ shall be called c_i, its centre C_i. The nine-point circle of $P_k\ P_l\ P_m$ shall be c_{ij}, its centre C_{ij}. The middle point of $P_l\ P_m$ shall be P_{ijk}. It appears then that $P_{ijk}\ P_{ikl}\ P_{kli}\ P_{lij}$ lie upon a circle of radius 1 whose centre lies half-way from P_m to the centre of the circumscribed circle Q_m. The points $P_{ij} P_{jk} P_{ki}$ are all at a distance unity from P_{ijk} and so lie on a circle of radius 1 through Q_m. This circle may, for the present be called c_{ijk}. Consider, now, the three circles $c_i\, c_j\, c_k$.

---

* Perhaps the first proof of this was given by Greiner: Ueber das Kreisviereck, *Grunert's Archiv der Math. und Phys.*, vol. 60, 1877.

c_i contains C_{ij}, C_{ki}, C_{il};
c_j contains C_{ij}, C_{jk}, C_{jl};
c_k contains C_{jk}, C_{ki}, C_{kl}.

But C_{il}, C_{jl}, C_{k}, are the vertices of a circular triangle, with the three following sides.:

c_{ijl} containing C_{il}, C_{ij}, C_{jl};
c_{jkl} containing C_{jl}, C_{jk}, C_{kl};
c_{kil} containing C_{kl}, C_{ki}, C_{il};

and the three circles $c_{ijl}\, c_{jkl}\, c_{kil}$ are concurrent in Q_m. Hence, by Theorem 2 the three circles c_i, c_j, c_k are concurrent, or all five circles c_i pass through a point. But all have the radius 1. Hence their centres lie on a circle of radius 1 about the point of concurrence as centre.

We have thus proved our theorem for $n = 4$ and $n = 5$. But this very proof is independent of the number 5, and shows equally well that if the theorem hold for $n - 2$ and $n - 1$ it holds for n also, the only modification being that c_{ijk} has now a natural instead of an artificial significance, and Q_m is replaced by P_{ijkl}, a definite point in every case where $n \geqq 6$. Our theorem is thus proved in its entirety.

We may easily obtain other systems of circles, similarly connected, by a dilatation away from the centre of the circumscribed circle with any chosen ratio. For instance if we dilate with the ratio 2, in the case where $n = 4$, we pass from the circle through the centres of four nine-point circles to that through four orthocentres, for the orthocentre is twice as far away from the centre of the circumscribed circle as is the centre of the nine-point circle.

It is now time to redeem our pledge to make some use of Theorem 3. Suppose that we have four concyclic points $P_1\, P_2\, P_3\, P_4$. They may be arranged in three different cyclic orders. The circle through P_i and P_j shall be called c_{ij}. The second intersection of c_{ij} and c_{jk} shall be $P_{ij,jk}$. We have twelve such points arranged as follows:

$P_{14,42}$, $P_{42,23}$, $P_{23,31}$, $P_{31,14}$ lying on c_2',
$P_{12,23}$, $P_{13,34}$, $P_{34,41}$, $P_{41,12}$ lying on c_3',
$P_{13,34}$, $P_{34,42}$, $P_{42,21}$, $P_{21,13}$ lying on c_4'.

Consider the sextic curve $(c_2' c_{12} c_{34})$. It passes through each ordinary intersection of the quartics $(c_{13} c_{24})$ and $(c_{14} c_{23})$, and has at each circular point at infinity a triple point, whereas each quartic has a double point there. We must then by Nöther's fundamental theorem have an identity of the sort

$$(c_2' c_{12} c_{34}) \equiv \phi_3(c_{13} c_{24}) + \phi_4(c_{14} c_{23}).$$

The curves ϕ_3 and ϕ_4 must be conics and each must pass once through each circular point. Hence they are circles.

Moreover ϕ_3 contains $P_{12,23}$, $P_{23,34}$, $P_{34,41}$, $P_{41,12}$ and so must be identical with c_3', and ϕ_4 for a like reason is identical with c_4'. Hence finally, the points common to c_3' and c_4' belong to c_2' also, that is to say:

II) *Four concyclic points may be arranged in three different cyclic orders, each giving rise to a cyclic order among four out of six circles of arbitrary size, which connect the given points in pairs. Successive circles in each cyclic order will intersect again in four concyclic points, and the three circles so determined are co-axial.*

It is natural to ask what analogues exist in three dimensions to the various theorems which we have here discussed. There are two rival claimants to the honor to be the three dimensional analogue of the nine-point circle, but neither seems to give rise to anything corresponding to Theorem 2. Theorem 1, on the other hand, may easily be generalized as follows:*

4) *Let a point be marked on each edge of a tetraedron, and let a sphere be passed through each vertex, and the marked points of the adjacent edges; these four spheres will be concurrent.*

This theorem may be proved in much the same way as was 1. There is an analogue to 3) also which has not, so far as I know, ever been published.

III) *Given five points upon a sphere arranged in cyclic order. Through each set of three successive points a sphere of arbitrary size shall be passed, thus determining five other points, each the remaining intersection of three successive spheres. These five points also lie upon a sphere.*

---

*This theorem was first proved by Roberts: On certain Tetraedra specially related to four Spheres meeting in a point, *Proceedings London Mathematical Society*, vol. 11, 1880. It was given implicitly much earlier by Miquel: Memoire de Geometrie, *Liouville's Journal*, vol. 11, 1845.

It is remarkable that whereas Theorem 3 comes immediately from Theorem 1 by an inversion, its analogue III does not seem to bear any close relation to 4. We find a proof on different lines as follows: The five given points shall be P_1, P_2, P_3, P_4, P_5, lying on s. The sphere through $P_iP_jP_k$ shall be called s_{ijk}; the remaining intersection of $s_{ijk}s_{jkl}s_{klm}$ shall be P'_k. Let us consider the following equation:

$$\lambda_1 s_{451} s_{123} + \lambda_2 s_{512} s_{234} + \lambda_3 s_{123} s_{345} + \lambda_4 s_{234} s_{451} + \lambda_5 s_{345} s_{512} = 0.$$

The quantities λ_i being constants, we see that we have here a cyclid through all ten of our points $P_i\, P_i'$. There is no linear dependence among the various terms, so that by varying the λ_i's we have really a four parameter family of surfaces. They will intersect s in a system of cyclics through five fixed points. Let us make use of two of our parameters to pass these curves through two other chosen points. We have still two free parameters, whereas the cyclics on any sphere through seven points contain an eighth point also, and form a one-parameter system, as we easily see by projecting them into cubics from one of their intersections. Hence, we may use our remaining parameters to make the cyclid include the sphere s entirely. The remainder of the cyclid will be a sphere through the five points P_i'.

There does not seem to be any simple three-dimensional analogue to II.

Cambridge, Mass.,
January, 1910.